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University and School Extension.

SPHERICAL TRIGONOMETRY.

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SPHERICAL TRIGONOMETRY.

THE SPHERICAL BLACKBOARD.

§ 1. THE student should construct his spherical triangles on a globe. He will thus get a good idea of the meaning of each problem, and the unknown parts of the triangle may be fairly well measured, and a good check against large errors in his calculations will be obtained.

A cylindrical cup, whose depth is half its diameter, or a hemispherical cup, should be provided with a flat rim graduated to degrees. To this cup should be fitted a slated globe, so it will turn easily in any direction.

The rim of the cup is the ruler used for drawing and measuring the arcs of great circles.

Dividers may also be used for laying off the lengths of arcs, their measures being taken from the globe or from the graduated rim of the cup.

An angle is measured by laying off on each of its including sides, or on those sides produced, $90°$; the arc joining these two points thus found will be the measure of the angle.

Two equal triangles may be made to coincide by direct superposition, and two symmetrical triangles by turning one of them inside out, and then superposing it on the other.

[A piece of tin-foil may be fitted to a sphere and figures cut out from it as they are cut out of paper in plane geometry to make this superposition.]

PROBLEMS OF CONSTRUCTION.

1. To draw one great circle perpendicular to another one.
2. To construct an angle at a given point equal to a given angle already marked out on the sphere.
3. To construct an angle at a given point equal to an angle of a given number of degrees.
4. Having a triangle given, to construct its polar triangle.
5. To construct a triangle having given in degrees :

   (*a.*) the three sides.
   (*b.*) two sides and the included angle.
   (*c.*) two sides and an angle opposite one of them.
   When are there two solutions? When one? When none?
   (*d.*) two angles and a side opposite one of them.
   (*e.*) two angles and the included side.
   (*f.*) the three angles.

6. To construct the triangles as above, having the given lines and angles laid down on the sphere.

MODEL OF A PORTION OF THE SPHERE.

§ 2. The relations between the sides and angles of a spherical triangle may be best obtained from a pasteboard model of a portion of the sphere.

The sides of a spherical triangle are the intersecting arcs which

planes passing through the centre of the sphere make with the surface of the sphere.

The angles of a spherical triangle are the diedral angles formed by the planes of its sides.

If a line be drawn in each of the two faces of a diedral angle perpendicular to its edge at a given point, the plane angle formed by these two lines is the measure of this diedral angle.

RIGHT-ANGLED SPHERICAL TRIANGLES.

CONSTRUCTION OF MODEL. DERIVATION OF FUNDAMENTAL FORMULAS.

§ 3. A model for this purpose may be made as follows: On a piece of pasteboard lay off from a point O, as a centre, the line O C equal to the assumed radius of the sphere, and describe the arc of a circle B A C B′ in pencil, O being the centre. Fig. 7.

Draw the several lines as in this figure. The angles A O B, A O C, C O B′ are to be of the same size as in the figure; but the radius O C may be taken of any

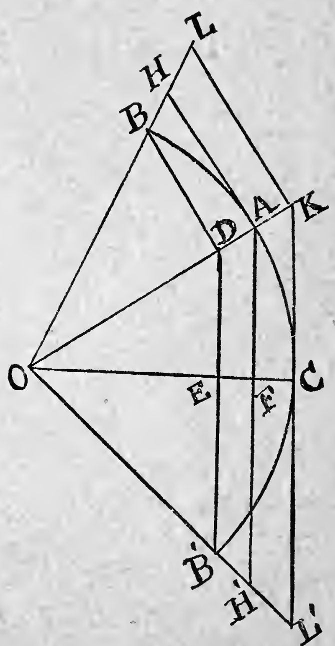

FIG. 7.

length, and the other lines, of course, must be taken proportional to O C.

Cut half through the pasteboard on the reverse side along the lines O C and O A, and cut out the entire figure O L K C L′ O. Bend the side O A H about O A till it is perpendicular to O A C, and bend O C B′ about O C till O B′ coincides with O B.

Call the arcs B′ C, A B, and A C respectively *a*, *c*, and *b*, and label them accordingly. These are the *sides* of the right spherical triangle; A is the right angle, C the angle at the base, and B the angle at the vertex. See Fig. 8.

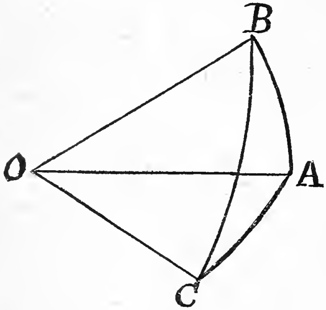

FIG. 8.

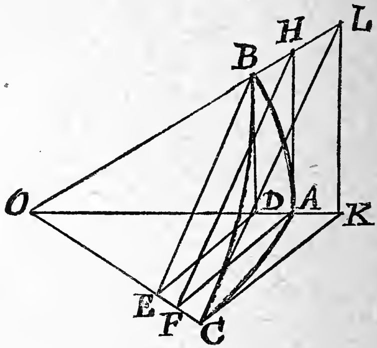

FIG. 9.

Construct also three plane right triangles from separate pieces of pasteboard.

1st. B D E, where the base and perpendicular are respectively equal to D E and B D as found above.

2d. H A F, having F A and A H as base and perpendicular.

3d. K L C, having K C and K L as base and perpendicular.

Fasten these three triangles securely in their places to the sides O B A and O A C of the model, leaving O C B′ to swing on O B′ as a hinge. The several planes D B E, H A F, and L K C will be perpendicular to the plane O A C and to the line O C, and the points and lines of the triangles will meet the points and lines of the model, which have corresponding letters. See Fig. 9.

The angles E, F, and C in these triangles are each equal to the angle C of the spherical triangle.

DERIVATION OF FORMULAS.

§ 4. In the plane right triangle D B E. Fig. 10.

$$\left.\begin{aligned} DB &= \sin c \\ EB' &= \sin a \end{aligned}\right\} \text{— the radius of the sphere, being unity.}$$

The angle D E B = angle C of the spherical triangle A B C.

$$\sin C = \frac{\sin c}{\sin a} \qquad (1)$$

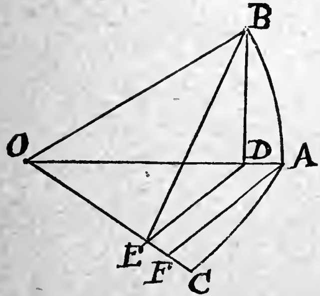

FIG. 10.

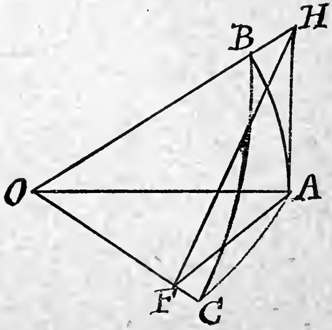

FIG. 11.

In the plane right triangle F A H. Fig. 11.

$$HA = \tan c$$
$$AF = \sin b$$

The angle H F A = angle C of the spherical triangle.

$$\tan C = \frac{\tan c}{\sin b} \qquad (2)$$

In the plane right triangle K C L. Fig. 12.

$$CL = \tan a$$
$$CK = \tan b$$

The angle K C L = angle C of the spherical triangle.

$$\cos C = \frac{\tan b}{\tan a} \qquad (3)$$

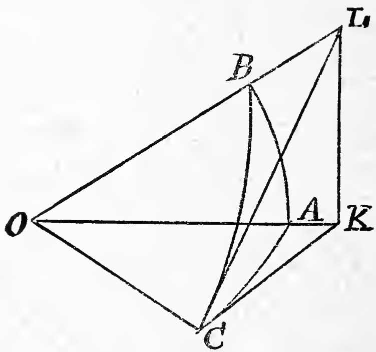

FIG. 12.

In the plane triangles O D E and A O F. Fig. 10.

$OF = \cos b$ $\qquad$ $OD = \cos c$

$OE = \cos a$ $\qquad$ $OA =$ unity $=$ radius of sphere.

$$\cos a = \cos b \cos c \qquad (4)$$

If the model were made with c for the base and B the base angle we could obtain the following relations:

$$\sin B = \frac{\sin b}{\sin a} \qquad (5)$$

$$\tan B = \frac{\tan b}{\sin c} \qquad (6)$$

$$\cos B = \frac{\tan c}{\tan a} \qquad (7)$$

By combining the above formulas we may obtain

$$\sin B = \frac{\cos C}{\cos c} \qquad (8)$$

$$\sin C = \frac{\cos B}{\cos b} \qquad (9)$$

$$\cos a = \cot C \cot B \qquad (10)$$

GROUPING OF FORMULAS.

§ 5. It will be seen that the above are all the possible combinations of the five parts of the right spherical triangle B, C, a, b, c, taken in sets of three, and therefore we have all the cases that can arise in their solution. For convenience of memory these ten formulas may be arranged in two groups.

1st. Those which involve tangents and cotangents.

2d. Those which involve only sines and cosines.

1ST GROUP.

$\cos a = \cot B \cot C$	or	$\sin (\text{co. } a)$	$= \tan (\text{co. B}) \tan (\text{co. C})$.
$\cos B = \cot a \tan c$	or	$\sin (\text{co. B})$	$= \tan (\text{co. } a) \tan c$.
$\cos C = \cot a \tan b$	or	$\sin (\text{co. C})$	$= \tan (\text{co. } a) \tan b$.
$\sin b = \tan c \cot C$	or	$\sin b$	$= \tan c \tan (\text{co. C})$.
$\sin c = \tan b \cot B$	or	$\sin c$	$= \tan b \tan (\text{co. B})$.

2D GROUP.

$\cos a = \cos b \cos c$	or	$\sin (\text{co. } a)$	$= \cos b \cos c$.
$\cos B = \sin C \cos b$	or	$\sin (\text{co.B})$	$= \cos (\text{co. C}) \cos b$.
$\cos C = \sin B \cos c$	or	$\sin (\text{co.C})$	$= \cos (\text{co. B}) \cos c$.
$\sin b = \sin a \sin B$	or	$\sin b$	$= \cos (\text{co. } a) \cos (\text{co. B})$.
$\sin c = \sin a \sin C$	or	$\sin c$	$= \cos (\text{co. } a) \cos (\text{co. C})$.

NAPIER'S RULES.

§ 6. If we draw a spherical triangle, making A the right angle, B and C the remaining angles, and a, b, and c the three sides, and then write comp. B, comp. a, comp. C for B, a, C, respectively, we shall see that by excluding the right angle A we may state as a rule covering the formulas in the first group :

I. *The sine of the middle part is equal to the product of the tangents of the adjacent parts.*

and for the second group—

II. *The sine of the middle part is equal to the product of the cosines of the opposite parts.*

It will help the student to avoid the confusion of these rules if he will observe that the words *ta*ngent and *a*djacent go together, and also that the words c*o*sine and *o*pposite go together.

RULE FOR THE SOLUTION OF RIGHT SPHERICAL TRIANGLES.

§ 7. By I. and II. *find directly from the two given parts each of the remaining parts.*

CHECK.—*Substitute the proper values in a formula containing the three required parts.*

EXAMPLES.

The student should select a sufficient number of examples from any text-book for solution, and also plot these examples on the globe, measuring the parts required in the solution.

PROBLEM.

Make the necessary computations for a sundial.

OBLIQUE-ANGLED SPHERICAL TRIANGLES.

EXERCISES ON THE GLOBE.

§ 8. Construct the following triangles from the given parts, and measure the remaining parts:

1. Given the three sides, 38°, 56°, 70°.
2. Given the three angles, 75°, 80°, 112°.
3. Given two sides and included angle, 80°, 74°, 60°.
4. Given two angles and a side opposite one of them, 60°, 80°, 94°.
5. Given two angles and the included side, 40°, 55°, 70°.

6. Given two sides and an angle opposite one of them, 85°, 65°, 50°. How many cases in this last?
7. Draw the polar triangle for each case.

MODEL FOR OBLIQUE-ANGLED TRIANGLE.

A model may be made to derive the fundamental formula of oblique-angled triangles as follows:

On a piece of pasteboard take O as a centre, with a radius O A, and describe an arc A B C D.

Draw O B, and produce it to meet the perpendicular A E erected to OA at A, and draw O C to meet the perpendicular D F erected to O D at D.

Cut out the piece of pasteboard O A E F D O.

Cut half through the pasteboard along O E and O F.

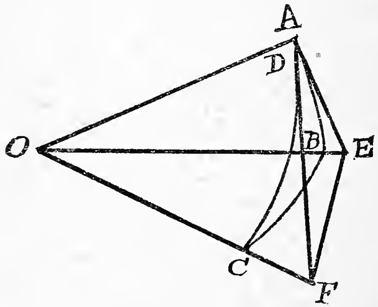

FIG. 13.

Bend the two outside triangles so that O D and O A will coincide, and fasten the pieces securely together, calling the radius OA unity, and the sides of the spherical triangle respectively opposite A, B, C,—*a*, *b*, and *c*.

F A E also measures the angle A of the spherical triangle.

$AE = \tan c \quad : \quad AF = \tan b$

$OE = \sec c \quad : \quad OF = \sec b.$

In the triangle A E F

$$\overline{FE}^2 = \tan^2 b + \tan^2 c - 2 \tan b \tan c \cos A.$$

In the triangle O E F

$$\overline{FE}^2 = \sec^2 b + \sec^2 c - 2 \sec b \sec^2 c \cos a.$$

Subtracting these two last equations and reducing, we have

$$\cos a = \cos b \cos c + \sin b \sin c \cos A. \qquad \text{(I.)}$$

By § 11 Plane Trigonometry, advancing the letters, we may get expressions for cos B and cos C.

§ 9. By the principle of Polar Triangles, if we substitute in the above formulas

$$\begin{array}{ll} A = 180 - a' & a = 180^\circ - A' \\ B = 180 - b' & b = 180^\circ - B' \\ C = 180 - c' & c = 180^\circ - C' \end{array}$$

we obtain, after reducing, and suppressing the accents :

$$-\cos A = \cos B \cos C - \sin B \sin C \cos a \qquad \text{(II.)}$$

etc., etc.

[The formula, before suppressing the accents, was true for all polar triangles, and since every possible triangle may be included in these polar triangles, the formula will hold generally.]

§ 10. In a spherical triangle A B C, let B and C be the angles at the base. The sides A B, BC, and C A are respectively c, a, b. Drop a perpendicular A D from A on the side B C.

By right triangle formulas we have

$$A\,D = \sin c \sin B = \sin b \sin C,$$

or, $$\frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \qquad \text{(III.)}$$

NOTE.—In this triangle the base a may be found by adding two segments B D and C D together. These segments may be found by the rules of right triangles.

Summary of Formulas for solving OBLIQUE ANGLED TRIANGLES without Logarithms.

§ 11. 1. Three sides Formulas I.
2. Three angles Formulas II.
3. Two sides and included angle I.
4. Two angles and included side II.
5. Two sides and opposite angle III. III. note.
6. Two angles and opposite side III. III. note.

SOLUTION OF OBLIQUE-ANGLED TRIANGLES

BY THE USE OF LOGARITHMS.

§ 12. From Formula (I.) $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$

Subtract each side of this equation from unity.

By Pl. Trig. $2 \sin^2 \tfrac{1}{2} A = 1 - \cos A$.

$$\sin^2 \tfrac{1}{2} A = \sqrt{\frac{\sin b \sin c - \cos a + \cos b \cos c}{2 \sin b \sin c}}$$

reducing

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\cos (b - c) - \cos a}{2 \sin b \sin c}}$$ [Prove this.]

By Pl. Trig. Formula 13.

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin \dfrac{a - b + c}{2} \sin \dfrac{a + b - c}{2}}{\sin b \sin c}}$$

Putting $\dfrac{a + b + c}{2} = s$

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}} \qquad \text{(IV.)}$$

§ 13. Add unity to each side of Formula (I.) and then prove in a similar way.

$$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin (s-a)}{\sin b \sin c}} \qquad \text{(V.)}$$

Dividing (IV.) by (V.)

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}} \qquad \text{(VI.)}$$

tan ½ B and tan ½ C may be found by advancing one letter throughout the formula. See § 12, Plane Trig.

§ 14. From II. derive in a similar way

$$\tan \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos (S - A)}{\cos (S - B) \cos (S - C)}} \qquad \text{(VII.)}$$

where $S = \dfrac{A + B + C}{2}$

§ 15. Dividing (VI.)

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}} \text{ by}$$

$$\tan \tfrac{1}{2} B = \sqrt{\frac{\sin (s - c) \sin (s - a)}{\sin s \sin (s - b)}}$$ (obtained by advancing one letter in VI.)

we have $\dfrac{\tan \frac{1}{2} A}{\tan \frac{1}{2} B} = \dfrac{\sin (s - b)}{\sin (s - a)}$

By composition and division

$$\frac{\tan \frac{1}{2} A + \tan \frac{1}{2} B}{\tan \frac{1}{2} A - \tan \frac{1}{2} B} = \frac{\sin (s - b) + \sin (s - a)}{\sin (s - b) - \sin (s - a)}$$

which reduces to

$$\frac{\sin \frac{1}{2} (A + B)}{\sin \frac{1}{2} (A - B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2} (a - b)} \qquad \text{(VIII.)}$$

Work out all the steps.

Multiplying tan ½ A by tan ½ B, values as above given, and reducing, we obtain

$$\frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} \qquad \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2}(a+b)} \qquad \text{(IX.)}$$

Work out all the steps.

§ 16. By using the principle of Polar triangles Formulas (VIII.) and (IX.) reduce to

$$\frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2}(A-B)} \qquad \text{(X.)}$$

and

$$\frac{\cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}(a-b)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2}(A+B)} \qquad \text{(XI.)}$$

Formulas (VIII.), (IX.), (X.) and (XI.) are called Napier's Analogies.

Formulas III., VI., VII., with the Napier's Analogies, are sufficient to solve all cases of spherical triangles by logarithms.

These are collected and renumbered in the following summary:

SUMMARY.

§ 17. [There are two additional formulas for each one of the following, and these may be obtained as indicated, § 12, Plane Trig.]

1. $$\frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

2. $$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}}$$

3. $$\tan \tfrac{1}{2} a = \sqrt{\frac{- \cos S \cos (S - A)}{\cos (S - B) \cos (S - C)}}$$

4. $$\frac{\sin \tfrac{1}{2} (A + B)}{\sin \tfrac{1}{2} (A - B)} = \frac{\tan \tfrac{1}{2} c}{\tan \tfrac{1}{2} (a - b)}$$

5. $$\frac{\cos \tfrac{1}{2} (A + B)}{\cos \tfrac{1}{2} (A - B)} = \frac{\tan \tfrac{1}{2} c}{\tan \tfrac{1}{2} (a + b)}$$

6. $$\frac{\sin \tfrac{1}{2} (a + b)}{\sin \tfrac{1}{2} (a - b)} = \frac{\cot \tfrac{1}{2} C}{\tan \tfrac{1}{2} (A - B)}$$

7. $$\frac{\cos \tfrac{1}{2} (a + b)}{\cos \tfrac{1}{2} (a - b)} = \frac{\cot \tfrac{1}{2} C}{\tan \tfrac{1}{2} (A + B)}$$

CASES.

§ 18. Given :

1. Two sides and angle opposite one of them	1, 4, 1.
2. Two angles and side opposite one of them	1, 4, 1.
3. Three sides	2, 2, 2.
4. Three angles	3, 3, 3.
5. Two sides and included angle	6, 7, 1.
6. Two angles and included side	4, 5, 1.

The proof in each case may be obtained by substituting the three quantities obtained in the complete solution in some formula.

EXAMPLES.

§ 19. 1. In the spherical triangle A B C let a, b, c be the sides respectively, it is required to find the remaining parts given.

(a) $A = 50°$, $b = 60°$, and $a = 40°$.

(b) $a = 50° 45' 20''$, $b = 69° 12' 40''$, and $A = 44° 22' 10''$. How many solutions are there?

(c) $A = 129° 05' 28''$, $B = 142° 12' 42''$, $C = 105° 08' 10''$.

(d) $a = 124° 53'$, $b = 31° 19'$, and $c = 171° 48' 42''$.

2. "Find the shortest distance in miles on the earth's surface from Berlin, latitude 52° 31′ 13″ N., longitude 13° 23′ 52″ E., to Alexandria, Egypt, latitude 31° 13′ N., longitude 29° 55′ E., the earth being considered a sphere whose radius is 3,962 miles."

3. Find the length of the shortest day in New Haven, Conn., latitude 42° 18′ N., it being assumed that the centre of the sun at rising or setting is 90° 50′ from the zenith, and the declination of the sun 23° 28′ N.

4. Find the time when twilight begins on the above day at New Haven, the sun being 18° below the horizon at that time.